Solutions

1. A system of linear equations has
   i. Infinitely many solutions,
   ii. Exactly one solution, or
   iii. No solutions.

If the system has no solutions we say it is ***inconsistent***, otherwise for (i) and (ii) we say the system is ***consistent***.

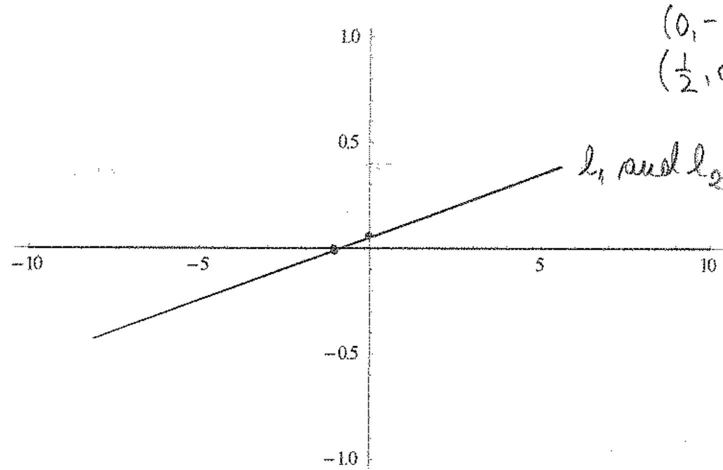
Using the equation indicated below, create a system of linear equations with two equations and two variables for each case. Solve the system algebraically and represent it graphically using lines.

Case i. ***Infinitely many solutions***

$$\begin{cases} 2x_1 - x_2 = 1 \\ 4x_1 - 2x_2 = 2 \end{cases} \Big\} \text{ multiply by } 2$$

Sol. $\begin{cases} x_1 = \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 \text{ free} \end{cases}$

$(0,-1)$
$(\frac{1}{2},0)$



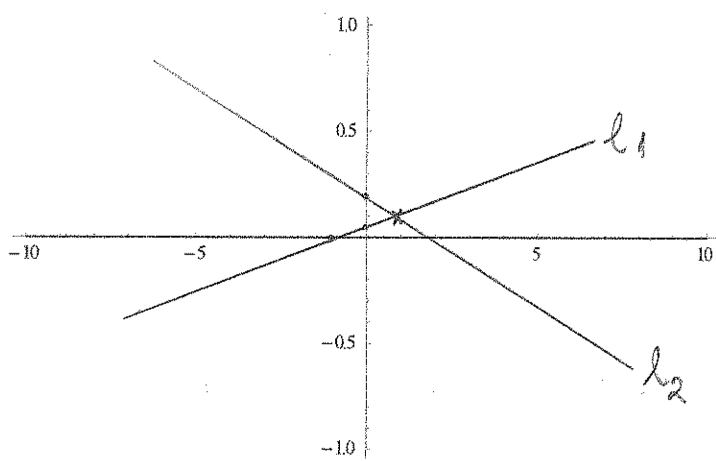
Case i. ***Exactly one solution***

$$\begin{cases} 2x_1 - x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$$

$\Rightarrow 3x_1 = 3 \Rightarrow x_1 = 1$

$\Rightarrow 1 + x_2 = 2 \Rightarrow x_2 = 1$

Sol. $(1,1)$



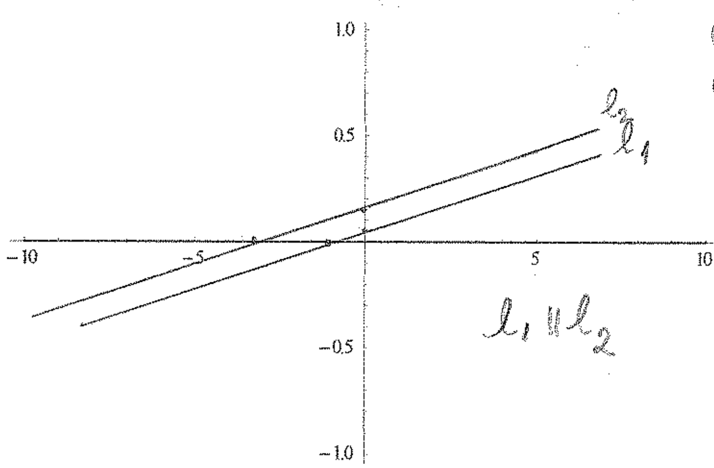
Case iii. ***No solutions***

$$\begin{cases} 2x_1 - x_2 = 1 \\ 2x_1 - x_2 = 3 \end{cases} \Big| \; -$$

$0 = 4$

$\Rightarrow$ no Sol.

$(0,-3)$
$(\frac{3}{2},0)$



2. Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$.

$$\begin{cases} x_1 + 2x_2 = -13 \\ 3x_1 - 2x_2 = 1 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & -13 \\ 3 & -2 & 1 \end{array}\right] \overset{-3r_1+r_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -13 \\ 0 & -8 & 40 \end{array}\right] \sim$$

$$\overset{-\frac{1}{8}r_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -13 \\ 0 & 1 & -5 \end{array}\right] \overset{-2r_2+r_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -5 \end{array}\right]$$

Sol $\begin{cases} x_1 = -3 \\ x_2 = -5 \end{cases}$ point of intersection is $(-3, -5)$.

3. Given an example of a matrix A with the indicated dimensions:

i. $A_{3\times2}$ $\begin{bmatrix} 1 & 6 \\ 0 & 2 \\ -2 & 1 \end{bmatrix}$

3 rows
2 columns

iii. $A_{3\times4}$ $\begin{bmatrix} 1 & 3 & 4 & 6 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

3 rows
4 columns

ii. $A_{1\times3}$ $\begin{bmatrix} 6 & 0 & 3 \end{bmatrix}$

1 row
3 columns

iv. $A_{4\times1}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

4 rows
1 column

4. Is $(3,4,-2)$ a solution for the following system?

$$\begin{cases} 5x_1 - x_2 + 2x_3 = 7 \\ -2x_1 + 6x_2 + 9x_3 = 0 \\ -7x_1 + 5x_2 - 3x_3 = -7 \end{cases}$$

$5 \cdot 3 - 4 + 2(-2) = 15 - 4 - 4 = 7$ ✓

$-2 \cdot 3 + 6 \cdot 4 + 9(-2) = -6 + 24 - 18 = 0$ ✓

$-7 \cdot 3 + 5 \cdot 4 - 3(-2) = -21 + 20 + 6 = 5$

$(3, 4, -2)$ is not a solution of the system

5. Solve the following systems of linear equations by following the indicated steps:

i. $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$

a) Write the augmented matrix.

b) $4r_1 + r_3$

c) $\frac{1}{2}r_2$

d) $3r_2 + r_3$

e) $4r_3 + r_2$

f) $-r_3 + r_1$

g) $2r_2 + r_1$

h) Record the solution.

i) Check your work by plugging in solution in the initial system of equations.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \overset{4r_1+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \overset{\frac{1}{2}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \overset{3r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\overset{4r_3+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{-r_3+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{2r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Sol. $\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$

ii. $\begin{cases} x_1 \quad - 2x_3 + 8x_4 = 12 \\ x_2 - 7x_3 \quad = -4 \\ 5x_3 - x_4 = 7 \\ x_3 + 3x_4 = -5 \end{cases}$

a) Write the augmented matrix.

b) $r_3 \leftrightarrow r_4$

c) $-5r_3 + r_4$

d) $-\frac{1}{16}r_4$

e) $-3r_4 + r_3$

f) $7r_3 + r_2$

g) $-8r_4 + r_1$

h) $2r_3 + r_1$

i) Record the solution.

j) Check your work by plugging in solution in the initial system of equations.

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & -7 & 0 & -4 \\ 0 & 0 & 5 & -1 & 7 \\ 0 & 0 & 1 & 3 & -5 \end{array}\right] \overset{r_3 \leftrightarrow r_4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & -7 & 0 & -4 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 5 & -1 & 7 \end{array}\right] \overset{-5r_3+r_4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & -7 & 0 & -4 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & -16 & 32 \end{array}\right]$$

$$\overset{-\frac{1}{16}r_4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & -7 & 0 & -4 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \overset{-3r_4+r_3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & -7 & 0 & -4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \overset{7r_3+r_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 12 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\overset{-8r_4+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 28 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \overset{2r_3+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Sol. $\begin{cases} x_1 = 30 \\ x_2 = 3 \\ x_3 = 1 \\ x_4 = -2 \end{cases}$ $(30, 3, 1, -2)$

6. Solve the following systems of equations by row-reducing the augmented matrix.

i. $\begin{cases} x_1 + 5x_2 = 7 \\ -2x_1 - 7x_2 = -5 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right] \overset{2r_1+r_2}{\sim} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right] \overset{\frac{1}{3}r_2}{\sim} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right] \sim$$

$$\overset{-5r_2+r_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right]$$

Sol. $\begin{cases} x_1 = -8 \\ x_2 = 3 \end{cases}$ $(-8, 3)$

ii. $\begin{cases} 3x_1 + 6x_2 = -3 \\ 5x_1 + 7x_2 = 10 \end{cases}$

$$\left[\begin{array}{cc|c} 3 & 6 & -3 \\ 5 & 7 & 10 \end{array}\right] \overset{\frac{1}{3}r_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 5 & 7 & 10 \end{array}\right] \overset{-5r_1+r_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 3 & 15 \end{array}\right] \sim$$

$$\overset{\frac{1}{3}r_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -5 \end{array}\right] \overset{-2r_2+r_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -5 \end{array}\right]$$

Sol. $\begin{cases} x_1 = 9 \\ x_2 = -5 \end{cases}$ $(9, -5)$

iii. $\begin{cases} x_1 - 3x_2 + 5x_3 - 2x_4 = 0 \\ x_2 + 8x_3 = -4 \\ 2x_3 = 3 \\ x_4 = 1 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & -3 & 5 & -2 & 0 \\ 0 & 1 & 8 & 0 & -4 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{2r_4+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 5 & 0 & 2 \\ 0 & 1 & 8 & 0 & -4 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{-4r_3+r_2}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 5 & 0 & 2 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim$$

$$\overset{\frac{1}{2}r_3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 5 & 0 & 2 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{3r_2+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & -46 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{-5r_3+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -107/2 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Sol. $\begin{cases} x_1 = \frac{-107}{2} \\ x_2 = -16 \\ x_3 = 3/2 \\ x_4 = 1 \end{cases}$ $\quad \left(\frac{-107}{2}, -16, \frac{3}{2}, 1\right)$

iv. $\begin{cases} x_2 + 5x_3 = -4 \\ x_1 + 4x_2 + 3x_3 = -2 \\ 2x_1 + 7x_2 + x_3 = -2 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right] \overset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{array}\right] \overset{-2r_1+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{array}\right] \sim$$

$$\overset{r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The system is inconsistent, since the last row gives us the equation $0=-2$. The solution set is empty.

v. $\begin{cases} x_1 \quad -3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ x_2 + 5x_3 = -2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \overset{-2r_1+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \overset{r_2 \leftrightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right] \sim$$

$$\overset{-2r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \overset{\frac{1}{5}r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{-5r_3+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\overset{3r_3+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

<u>Sol.</u> $\begin{cases} x_1 = 5 \\ x_2 = 3 \\ x_3 = -1 \end{cases}$ $(5, 3, -1)$

7. Mark each statement as ***True*** or ***False*** and justify your answer.

False i. A 5×6 matrix has six rows. → 5 rows

False ii. The solutions set of a linear system involving variables $x_1, x_2, \ldots x_n$ is a list of numbers $(s_1, s_2, \ldots s_n)$ that make each equation in the system a true statement when the values $s_1, s_2, \ldots s_n$ are substituted for $x_1, x_2, \ldots x_n$, respectively. → statement does not apply to systems with ∞-many solutions

False iii. Two matrices are row equivalent if they have the same number of rows. → only if you can get from A_1 to A_2 by row reduction operations

True iv. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

False v. Two equivalent linear systems can have different solution sets. → same solution set

True vi. A consistent system of linear equations has one or more solutions. → one or ∞-many solutions

True vii. Every elementary row operation is reversible.

8. The following augmented matrices have been transformed by row operations into echelon form. Determine if the corresponding linear systems are consistent (no further row reduction necessary).

i. $\left[\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

omits row $[0\ 0\ 0 \mid \blacksquare]$

$\Rightarrow$ system is consistent

ii. $\left[\begin{array}{cccc|c} 1 & 7 & 3 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$

$x_2 + x_4 = 0 \Rightarrow x_2 = 0$

$3x_3 = 0 \Rightarrow x_3 = 0$

$2x_4 = 0 \Rightarrow x_4 = 0$

omits row $[0\,0\,0\,0 \mid \blacksquare]$

$\Rightarrow$ system is consistent

iii. $\left[\begin{array}{ccc|c} 2 & 6 & 0 & -1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

is equivalent to $0 = 1$ System is inconsistent

9. Find the elementary row operation that transforms the first matrix into the second, and then find the reverse operation that transforms the second matrix into the first.

i. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix}$$

ii. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix} \overset{-\frac{1}{5}r_3}{\sim} \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix} \overset{-5r_3}{\sim} \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix}$$

iii. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix} \overset{-4r_1+r_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix} \overset{4r_1+r_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}$$

10. Determine which matrices are in reduced echelon form, which ones are only in echelon form, and which ones are in neither form.

i. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ reduced echelon form

iii. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ echelon form (not reduced)

ii. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ → row of zeros above row with non-zero entries ⇒ not in echelon form

iv. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ echelon form (not reduced)

11. Row reduce the following matrices to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

i. $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \overset{-2r_1+r_2}{\sim} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \overset{-3r_1+r_3}{\sim} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix}$

$$\underset{-\frac{1}{3}r_3}{\overset{-\frac{1}{2}r_2}{\sim}} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \overset{-r_2+r_3}{\sim} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{-4r_2+r_1}{\sim} \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns 1 and 3

ii. $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \overset{-2r_1+r_2}{\sim} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 4 & 5 & 4 & 2 \end{bmatrix} \overset{-4r_1+r_3}{\sim} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \sim$

$\overset{-\frac{1}{3}r_2}{\underset{-\frac{1}{3}r_3}{\sim}} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 6 \end{bmatrix} \overset{r_2 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \overset{-4r_3+r_2}{\sim} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim$

$\overset{-4r_3+r_1}{\sim} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \overset{-2r_2+r_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Pivot columns 1, 2 and 3.

12. Find the general solutions of the systems whose augmented matrices are given below:

i. $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{array}\right] \overset{3r_1+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & -2 & 0 & -6 \end{array}\right] \overset{-\frac{1}{2}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 3 \end{array}\right] \sim$

$\overset{3r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \end{array}\right] \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 3 \end{cases}$ The variable x_3 is free.

Sol. $\begin{cases} x_1 = 4 \\ x_2 = 3 \\ x_3 \text{ free} \end{cases}$ Note that leading variables do not depend on free variable.

ii. $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{array}\right] \overset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 4 & -6 \\ 0 & 1 & -2 & 3 \end{array}\right] \overset{3r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{array}\right]$

$\Rightarrow \begin{cases} x_1 - 2x_3 = 3 \\ x_2 - 2x_3 = 3 \end{cases}$ Sol. $\begin{cases} x_1 = 3 + 2x_3 \\ x_2 = 3 + 2x_3 \\ x_3 \text{ free} \end{cases}$

iii. $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{array}\right] \overset{-3r_1+r_2}{\sim} \left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & -4 & 8 & 0 \end{array}\right] \overset{-2r_1+r_3}{\sim} \left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\overset{\frac{1}{3}r_1}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

<u>Sol.</u> $\begin{cases} x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$

iv. $\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\Rightarrow$ $0 = 1$ $\Rightarrow$ system is inconsistent

System has no solution.

13. Mark each statement as ***True*** or ***False*** and justify your answer.

False i. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations. unique

echelon forms may be different

True ii. A leading variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

False iii. If one row in an echelon form of an augmented matrix is $\begin{bmatrix}0 & 0 & 0 & 5 & 0\end{bmatrix}$, then the associated linear system is inconsistent. gives equation $5x_4 = 0 \Rightarrow x_4 = 0$

True iv. The reduced echelon form of a matrix is unique.

False v. Whenever a system has free variables, the solution set contains infinitely many solutions.

$\rightarrow$ may still produce row $[0\ 0 \cdots 0 \mid 2]$ and make system inconsistent.

14. The following exercises represent a matrix using a schematic notation where entries denoted ■ may have any *nonzero* value, and entries denoted * may have *any* value (including 0).

Suppose that the following matrices each represent the echelon form of the augmented matrix of a system of linear equations. For each matrix, determine if the system is consistent. If the system is consistent, determine if the solution is unique. **You must justify your answer.**

i. $\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{array}\right]$

(a) No Row $[0\ 0\ 0 \mid \blacksquare] \Rightarrow$ consistent by Theorem 2

(b) pivot in each coefficient column $\Rightarrow$ unique solution by Theorem 2

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ii. $\left[\begin{array}{cccc|c} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{array}\right]$

(a) contains row $[0\ 0\ 0\ 0 \mid \blacksquare]$

$\Rightarrow$ inconsistent

(b) by Theorem 2

(or because ~~this~~ means $0 = \blacksquare$)

zero NOT zero

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iii. $\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$

not pivot column

(a) No row $[0\ 0\ 0 \mid \blacksquare] \Rightarrow$ consistent

(b) coefficient column without pivot

$\Rightarrow \infty$-many solutions

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iv. $\left[\begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & 0 \end{array}\right]$

Not pivot column

(a) No row $[0\ 0\ 0\ 0 \mid \blacksquare] \Rightarrow$ consistent

(b) coefficient column w/o pivot

$\Rightarrow \infty$-many solutions

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15. In the following exercises, determine the value of h so that the matrix is the augmented matrix of a consistent system of linear equations.

i. $\left[\begin{array}{cc|c} 2 & -3 & h \\ -6 & 9 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -3 & h \\ 0 & 0 & 5+3h \end{array}\right] \; r_2 + 3r_1$

Consistent $\Leftrightarrow$ <u>OMITS</u> row $[0\ 0 \mid \blacksquare]$

$\Leftrightarrow 5+3h=0$

$\Leftrightarrow h = -5/3$

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ii. $\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 5 & h & -7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{array}\right] \; r_2 - 5r_1$

Consistent $\Leftrightarrow$ <u>OMITS</u> row $[0\ 0 \mid \blacksquare]$

$\Leftrightarrow h+15 \neq 0$

$\Leftrightarrow h = -15$